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Knotted pictures of entangled states

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Abstract

From the comparison of the correlation tensor in the theory of quantum networks and the Alexander relation matrix in the theory of knot crystals we find that there is a one-to-one correspondence between four Bell bases and four oriented links of the linkage 4_1 in knot theory. Meanwhile, we show that the inversion relations under the action of Pauli matrices are the same for the four Bell bases and their corresponding links. Similarly we show that there is also a one-to-one correspondence between 2^m GHZ states of m nodes and 2^m oriented links of m components.

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1. Introduction

Since the pioneering work of Einstein *et al* [1], the concept of quantum entanglement has become more and more important in quantum mechanics [2–11]; it plays an essential role in quantum teleportation, quantum computation and the modern understanding of quantum phenomena. In this paper we shall study this problem from the point of view of knot crystals [12–14]. In the theory of quantum networks [15], the criterion of entanglement is the existence of nonzero elements of the correlation tensor [15]. Hence the correlation tensor plays a central role for studying the entangled state in quantum mechanics. On the other hand we find that the Alexander relation matrix [12] plays a central role in studying the entanglement of an oriented link with m components. From the comparison of the correlation tensor in the theory of quantum networks and the Alexander relation matrix in the theory of knot crystals we find that there is a one-to-one correspondence between four Bell bases and four oriented links of the linkage 4_1 in knot theory. Meanwhile, we show that the inversion relations under the action of Pauli matrices [16] are the same for the four Bell bases and their corresponding links. From the above arguments we have sufficient reasons to say that such correspondence is convincing and not accidental. Similarly we find that there is also a one-to-one correspondence between 2^m GHZ states of m nodes and 2^m oriented links of m components, and point out that the inversion relations under the action of Pauli matrices are the same for the 2^m GHZ states of

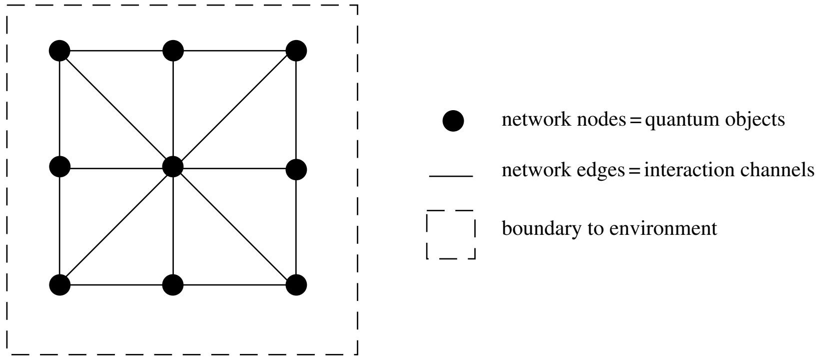

Figure 1. Quantum network.

m nodes and their corresponding 2^m oriented links. In section 2 we shall briefly introduce the correlation tensor and the covariant correlation tensor for a quantum network of two nodes. In section 3 we shall briefly introduce fundamental notions of knot theory and then derive the Alexander relation matrix for four oriented links of two components with four crossings and show that these links correspond to four Bell bases for the quantum network of two nodes; furthermore, we show that the inversion relations under the action of Pauli matrices are the same for the four Bell bases and their corresponding links. In section 4 we discuss GHZ states in general. In section 5 we discuss a quantum network of three nodes and generalize our results to a quantum network of m ($m \geqslant 4$) nodes.

2. Correlation tensor and covariance correlation tensor

A quantum network is a system consisting of N subsystems ('nodes') interacting with each other in n-dimensional Hilbert space. The subsystems of the network are special quantum objects denoted as 'network nodes'. Schematically this is represented by the diagram shown in figure 1; the interaction channels are represented by network edges.

The generating operators of $SU(n)$ are denoted by $\hat{\lambda}_j$. Let ν be the ordinal numbers of the nodes and s be the dimensionality of the generating operator $\hat{\lambda}_j$, where $s = n^2 - 1$. In the case of a single node, i.e. $\nu = 1$ only, in terms of the generating operators, the density operator $\hat{\rho}$ is [13]

$$\hat{\rho} = \frac{1}{n}\hat{1} + \frac{1}{2}\sum_{j=1}^{s}\lambda_j\hat{\lambda}_j \tag{1}$$

where λ_j is the expectation value of $\hat{\lambda}_j$, i.e.

$$\lambda_j = \langle\hat{\lambda}_j\rangle = \mathrm{Tr}\{\hat{\rho}\hat{\lambda}_j\}. \tag{2}$$

In the case $n = 2$, $s = n^2 - 1 = 3$, $\hat{\lambda}_j$ is just the Pauli operator $\hat{\sigma}_j$, whereas

$$\lambda_j = \langle\hat{\lambda}_j\rangle = \langle\hat{\sigma}_j\rangle = P_j \tag{3}$$

where P_j is a component of the polarization vector, $j = x, y, z$. Substituting (3) into (1) we obtain the density matrix:

$$\rho = \tfrac{1}{2}\begin{pmatrix} 1 + P_z & P_x - \mathrm{i}P_y \\ P_x + \mathrm{i}P_y & 1 - P_z \end{pmatrix}. \tag{4}$$

In the case of two nodes, $\nu = 1, 2$, the density operator $\hat{\rho}$ is [13]

$$\hat{\rho} = \frac{1}{n_1 n_2}\hat{1} + \frac{1}{2n_2}\sum_{j=1}^{s_1}\lambda_j(1)[\hat{\lambda}_j(1)\otimes\hat{1}(2)] + \frac{1}{2n_1}\sum_{k=1}^{s_2}\lambda_k(2)[\hat{1}(1)\otimes\hat{\lambda}_k(2)] + \frac{1}{4}\sum_{j.k=1}^{s_1,s_2}K_{jk}(1,2)[\hat{\lambda}_j(1)\otimes\hat{\lambda}_k(2)] \tag{5}$$

where $s_1 = n_1^2 - 1$ and $s_2 = n_2^2 - 1$. The correlation tensor $K_{jk}(1, 2)$ is defined as

$$K_{jk}(1,2) = \langle\hat{\lambda}_j(1)\hat{\lambda}_k(2)\rangle = \mathrm{Tr}\{\hat{\rho}[\hat{\lambda}_j(1)\otimes\hat{\lambda}_k(2)]\}. \tag{6}$$

The covariance correlation tensor $M_{jk}(1, 2)$ is introduced as

$$M_{jk}(1,2) = K_{jk}(1,2) - \lambda_j(1)\lambda_k(2). \tag{7}$$

Equations (6) and (7) are the definitions of the correlation tensor and covariant correlation tensor respectively. In terms of the covariant correlation tensor (5) can be written as

$$\hat{\rho} = \hat{\rho}(1)\otimes\hat{\rho}(2) + \frac{1}{4}\sum_{j.k=1}^{s_1,s_2}M_{jk}(1,2)[\hat{\lambda}_j(1)\otimes\hat{\lambda}_k(2)]. \tag{8}$$

When $M_{jk}(1, 2) = 0$ (for all possible j and k), (8) becomes the product state $\hat{\rho} = \hat{\rho}(1)\otimes\hat{\rho}(2)$, that is, there is no entanglement, i.e. no inter-node coherence. Hence the tensor $M_{jk}(1, 2)$ allows the term entanglement to be defined in a precise way. The criterion of an entangled state for a composite system consisting of two nodes is that $M_{jk}(1, 2) = 0$ is not true for all possible j and k, i.e. there exist nonzero elements of the covariance correlation tensor $M_{jk}(1, 2)$.

Now let us discuss the system consisting of two spin-$\frac{1}{2}$ particles. For this system, there are four well known Bell bases:

$$\begin{aligned}|\Psi^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle)\\ |\Phi^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle \pm |\downarrow\downarrow\rangle).\end{aligned} \tag{9}$$

Calculating the density matrix of the Bell bases and using (8) we obtain the corresponding covariance correlation tensors as follows:

$$\begin{aligned}M(\Psi^+) &= \begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix} \qquad M(\Psi^-) = \begin{pmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}\\ M(\Phi^+) &= \begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix} \qquad M(\Phi^-) = \begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}.\end{aligned} \tag{10}$$

Since there are nonzero elements in (10), Bell bases are entangled states.

Now let us discuss the inversion relations of Bell bases under the action of the Pauli operator [14]:

$$\sigma_1 = \begin{pmatrix}0&1\\1&0\end{pmatrix} = \sigma_x \qquad \sigma_2 = \begin{pmatrix}0&1\\-1&0\end{pmatrix} = \mathrm{i}\sigma_y \qquad \sigma_3 = \begin{pmatrix}1&0\\0&-1\end{pmatrix} = \sigma_z. \tag{11}$$

There are inversions of phase and spin, where σ_1 only causes inversion of spin and σ_3 only causes inversion of phase, but σ_2 can simultaneously cause inversion of phase and spin. The inversion relationship under the action of the Pauli operator is shown in figure 2.

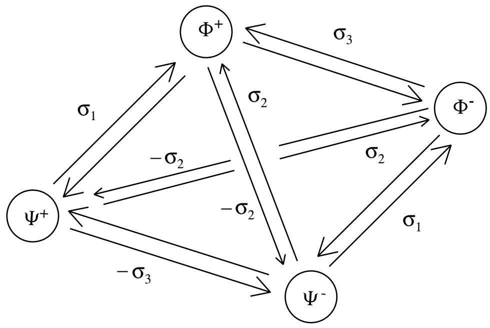

Figure 2. Inversion relationship of Bell bases.

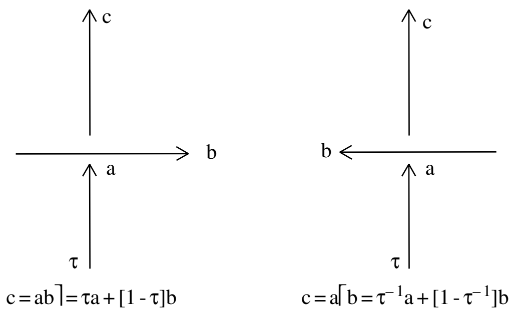

Figure 3. Two modes of crossing.

3. Alexander relation matrix and inversion relations

Firstly we briefly introduce fundamental notions of knot theory. A knot is a closed piece of string without loose ends; the length, thickness and precise shape are of no interest. A link is a collection of two or more knots, or components, which may or may not be physically intertwined. Knots and links can be projected onto a plane and thus represented by planar diagram. A knot, or a link, is oriented if its loops are directed; otherwise, the knot is unoriented. There are two kinds of oriented crossing. For every crossing, one oriented line segment situates over the crossing and the other oriented line segment situates under the crossing: when the direction from the overcrossing oriented line segment to the undercrossing oriented line segment is anticlockwise, the crossing is called a positive crossing; otherwise, i.e. clockwise, the crossing is called a negative crossing. For example, the left crossing shown in figure 3 is a positive crossing, whereas the right crossing shown in the same figure is a negative crossing.

Now we introduce Alexander relation matrices (ARMs) and derive them for a two-component oriented link with four crossings, where the two-component unoriented link with four crossings in knot theory is denoted by 4_1, where 4 denotes the number of crossings and the subscript denotes the order of such links [12]. For example, there are three types for a two-component unoriented link with six crossings: they are 6_1, 6_2, 6_3; thus the order of a link tells us which link of the class with known crossings it is. For a two-component unoriented link with four crossings, there is only one type; this is 4_1. Then we shall show that their nonzero submatrices have a one-to-one correspondence with the covariance correlation tensor of a system consisting of two nodes. Hence we can give the knotted picture of four Bell bases.

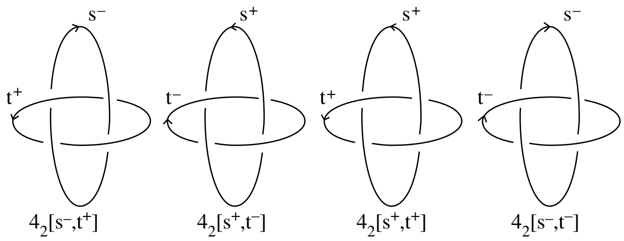

Figure 4. Four kinds of oriented link for a two-component link.

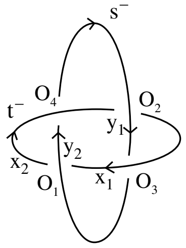

Figure 5. Diagram showing $4_2(s^-, t^-)$.

There are two modes of crossing, the right cross $b\rceil$ and the left cross $\lceil b$; they will serve to encode the two oriented types of diagrammatic crossing as shown in figure 3.

As illustrated in figure 3, the arc c emanating from an undercrossing is regarded as the result of the overcrossing arc b, acting ($b\rceil$ or $\lceil b$) on the incoming undercrossing arc a. Thus $c = a\ b\rceil$ corresponds to a right cross whereas $c = a\lceil b$ corresponds to a left cross. $a\ b\rceil$ and $a\lceil b$ are defined by the following equations:

$$ab\rceil = \tau a + (1 - \tau)b \qquad a\lceil b = \tau^{-1}a + (1 - \tau^{-1})b \tag{12}$$

where $a, b \in M$, M is any module over the ring $Z(\tau, \tau^{-1})$ and τ is the ring parameter.

For the convenience of discussion of a system consisting of more than two nodes, we use the notations $4_2(s^\pm, t^\pm)$ to represent the four oriented links, where the number 4 denotes that the number of oriented links is equal to four, the subscript 2 denotes that the number of components is equal to two and $t^\pm$ and $s^\pm$ denote the ring parameters of the four oriented components respectively: the positive sign denotes the anticlockwise direction of the corresponding component, whereas the negative sign denotes the clockwise direction of the corresponding component. Thus there are four kinds of oriented link for a two-component link: they are $4_2(s^+, t^+)$, $4_2(s^-, t^-)$, $4_2(s^-, t^+)$, $4_2(s^+, t^-)$, which are shown in figure 4.

Now we shall use (12) to derive the ARMs of the four oriented links $4_2(s^\pm, t^\pm)$. From the requirement that the Alexander polynomial of 4_1 is $P_A(s, t) = s + t$, irrespective of the directions of two components [12], i.e. the Alexander polynomial of the four oriented links $4_2(s^\pm, t^\pm)$ must have the same value as $P_A(s, t) = s + t$, we find that

$$s^+ = \frac{1}{s^-} = s \qquad t^- = \frac{1}{t^+} = t. \tag{13}$$

3.1. ARM of $4_2(s^-, t^-)$

$4_2(s^-, t^-)$ is shown in figure 5.

Using (12) for the four crossings we obtain

$$\begin{aligned}
&\text{For crossing } O_1 : x_2 = x_1 y_2\rceil = t^- x_1 + (1-t^-)y_2\\
&\text{For crossing } O_2 : x_1 = x_2 y_1\rceil = t^- x_2 + (1-t^-)y_1\\
&\text{For crossing } O_3 : y_2 = y_1 x_1\rceil = s^- y_1 + (1-s^-)x_1\\
&\text{For crossing } O_4 : y_1 = y_2 x_2\rceil = s^- y_2 + (1-s^-)x_2.
\end{aligned} \tag{14}$$

Using (13) to arrange (14) we obtain

$$\begin{array}{ll}
-tx_1 + x_2 + (t-1)y_2 = 0 & x_1 - tx_2 + (t-1)y_1 = 0\\
(s^{-1}-1)x_1 - s^{-1}y_1 + y_2 = 0 & (s^{-1}-1)x_2 + y_1 - s^{-1}y_2 = 0.
\end{array} \tag{15}$$

The coefficient matrix of (15) after eliminating negative powers is

$$A^{--}(s,t) = \begin{pmatrix} A_{11}^{--} & A_{12}^{--} \\ A_{21}^{--} & A_{22}^{--} \end{pmatrix} \tag{16}$$

where the matrices A_{ij}^{--} are all 2×2 matrices:

$$\begin{array}{ll}
A_{11}^{--} = \begin{pmatrix} -t & 1 \\ 1 & -t \end{pmatrix} & A_{12}^{--} = \begin{pmatrix} 0 & t-1 \\ t-1 & 0 \end{pmatrix}\\
A_{21}^{--} = \begin{pmatrix} 1-s & 0 \\ 0 & 1-s \end{pmatrix} & A_{22}^{--} = \begin{pmatrix} s & -1 \\ -1 & s \end{pmatrix}.
\end{array} \tag{17}$$

The 4×4 matrix (16) is the ARM of $4_2(s^-, t^-)$; from (16) we can easily obtain the nonzero 3×3 submatrix:

$$A^{--} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{18}$$

3.2. *ARM of* $4_2(s^+, t^-)$

Using the above method we readily obtain the ARM of $4_2(s^+, t^-)$:

$$A^{+-}(s,t) = \begin{pmatrix} A_{11}^{+-} & A_{12}^{+-} \\ A_{21}^{+-} & A_{22}^{+-} \end{pmatrix} \tag{19}$$

where

$$A_{ij}^{+-} = \sigma_2 A_{ij}^{--}. \tag{20}$$

The nonzero submatrix of (19) is

$$A^{+-} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{21}$$

3.3. *ARM of* $4_2(s^+, t^+)$

Similarly we obtain the ARM of $4_2(s^+, t^+)$:

$$A^{++}(s,t) = \begin{pmatrix} A_{11}^{++} & A_{12}^{++} \\ A_{21}^{++} & A_{22}^{++} \end{pmatrix} \tag{22}$$

where

$$A_{ij}^{++} = \sigma_3 A_{ij}^{--}. \tag{23}$$

The nonzero submatrix of (22) is

$$A^{++} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{24}$$

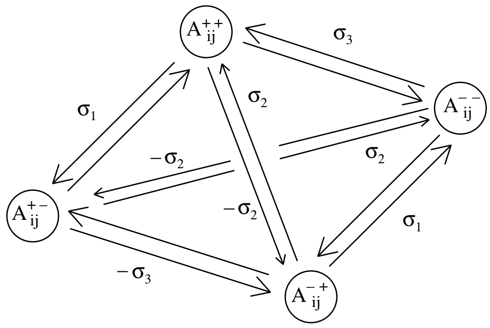

Figure 6. Inversion relationships of four oriented links.

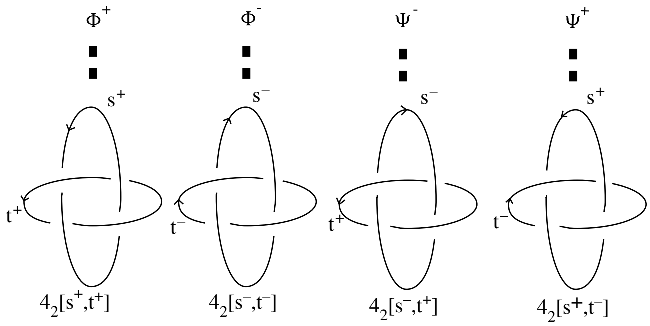

Figure 7. Correspondence between Bell bases and oriented links.

3.4. *ARM of* $4_2(s^-, t^+)$

Finally we obtain the ARM of $4_2(s^-, t^+)$:

$$A^{-+}(s, t) = \begin{pmatrix} A_{11}^{-+} & A_{12}^{-+} \\ A_{21}^{-+} & A_{22}^{-+} \end{pmatrix} \tag{25}$$

where

$$A_{ij}^{-+} = \sigma_1 A_{ij}^{--}. \tag{26}$$

The nonzero submatrix of (25) is

$$A^{-+} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{27}$$

3.5. *Correspondence between four Bell bases and four oriented links of* 4_1 *in knot theory*

From (20), (23) and (26) we know that there are inversion relationships between the ARMs of four oriented links under the action of the Pauli operator; these relations are shown in figure 6.

Comparing figure 6 with 2 and meanwhile comparing (18), (21), (24) and (27) with (10), obviously we have one-to-one correspondence between four Bell bases and four oriented links $4_2(s^+, t^+)$, $4_2(s^-, t^-)$, $4_2(s^-, t^+)$, $4_2(s^+, t^-)$. This correspondence is shown in figure 7.

4. GHZ states

Now we shall generalize our previous results for a quantum system of two nodes to a quantum system of m ($m \geqslant 2$) nodes. In the case of a system of two nodes, we know that there are four product states, which are $|\downarrow\downarrow\rangle = |1\rangle, |\downarrow\uparrow\rangle = |2\rangle, |\uparrow\downarrow\rangle = |3\rangle, |\uparrow\uparrow\rangle = |4\rangle$. From these four product states we construct four Bell bases, which are $|\Phi^{\pm}\rangle = (1/\sqrt{2})(|4\rangle \pm |1\rangle)$ and $|\Psi^{\pm}\rangle = (1/\sqrt{2})(|3\rangle \pm |2\rangle)$. Actually, Bell bases are GHZ states for $m = 2$. In the case of three nodes, there are eight product states, which are $|\downarrow\downarrow\downarrow\rangle = |1\rangle, |\downarrow\downarrow\uparrow\rangle = |2\rangle, |\downarrow\uparrow\downarrow\rangle = |3\rangle$, $|\downarrow\uparrow\uparrow\rangle = |4\rangle, |\uparrow\downarrow\downarrow\rangle = |5\rangle, |\uparrow\downarrow\uparrow\rangle = |6\rangle, |\uparrow\uparrow\downarrow\rangle = |7\rangle, |\uparrow\uparrow\uparrow\rangle = |8\rangle$. From these eight product states we can construct eight GHZ states for $m = 3$:

$$
\begin{aligned}
|\Phi_1^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|8\rangle \pm |1\rangle) \qquad & |\Phi_2^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|7\rangle \pm |2\rangle) \\
|\Phi_3^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|6\rangle \pm |3\rangle) \qquad & |\Phi_4^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|5\rangle \pm |4\rangle).
\end{aligned} \tag{28}
$$

The general GHZ states for $m \geqslant 2$ states are defined as

$$
|\Phi_j^{\pm}\rangle = \frac{1}{\sqrt{2}}(|2^m - j + 1\rangle \pm |j\rangle) \qquad j = 1, 2, \ldots, 2^{m-1}. \tag{29}
$$

The eight GHZ states for $m = 3$ have a similar inversion relationship as the case for $m = 2$; they are

$$
\begin{aligned}
&\sigma_3\Phi_j^{+} = \Phi_j^{-} \qquad && \sigma_3\Phi_j^{-} = \Phi_j^{+} \qquad && j = 1, 2, 3, 4 \\
&\sigma_1\Phi_1^{\pm} = \Phi_2^{\pm} && \sigma_1\Phi_2^{\pm} = \Phi_1^{\pm} && \sigma_2\Phi_1^{\pm} = \Phi_2^{\mp} \\
&\sigma_2\Phi_1^{\mp} = \Phi_2^{\pm} && \sigma_2\Phi_3^{\pm} = \Phi_4^{\mp} && \sigma_2\Phi_4^{\mp} = \Phi_3^{\pm}.
\end{aligned} \tag{30}
$$

Hence, as in the case for $m = 2$, σ_3 only causes inversion of phase and σ_1 only causes inversion of spin, but σ_2 can simultaneously cause inversion of phase and spin. The above statement can be generalized to GHZ states for any value of $m \geqslant 4$.

5. One-to-one correspondence between GHZ states and oriented links

In the case of a quantum network of three nodes, there are eight GHZ states: we shall show that they correspond to eight oriented links $8_3(s^{\pm}, t^{\pm}, u^{\pm})$, where the number 8 denotes that the number of oriented links is equal to eight, the subscript 3 denotes that the number of components is equal to three and s, t and u denote the ring parameters of the three components respectively; the positive sign denotes the anticlockwise direction of the corresponding component, whereas the negative sign denotes the clockwise direction of the corresponding component. Since any two components of the link have four crossings, there are 12 crossings in this case. The eight kinds of oriented link for a three-component link with 12 crossings are $8_3(s^+, t^-, u^-)$, $8_3(s^-, t^+, u^+)$, $8_3(s^+, t^+, u^+)$, $8_3(s^-, t^-, u^-)$, $8_3(s^+, t^+, u^-)$, $8_3(s^-, t^-, u^+)$, $8_3(s^+, t^-, u^+)$, $8_3(s^-, t^+, u^-)$. For example, the oriented link $8_3(s^+, t^-, u^-)$ is one of the eight oriented links with three components; the direction of the first component knot is anticlockwise, whereas the directions of the second and third component knots are clockwise. Using the same method as in section 3, of course the calculation is very tedious; we found that there is a one-to-one correspondence between eight GHZ states and eight oriented 8_3 links:

$$\begin{aligned} \Phi_1^+ &\longleftrightarrow 8_3(s^+, t^+, u^+) & \Phi_1^- &\longleftrightarrow 8_3(s^-, t^-, u^-) \\ \Phi_2^+ &\longleftrightarrow 8_3(s^+, t^+, u^-) & \Phi_2^- &\longleftrightarrow 8_3(s^-, t^-, u^+) \\ \Phi_3^+ &\longleftrightarrow 8_3(s^+, t^-, u^+) & \Phi_3^- &\longleftrightarrow 8_3(s^-, t^+, u^-) \\ \Phi_4^+ &\longleftrightarrow 8_3(s^+, t^-, u^-) & \Phi_4^- &\longleftrightarrow 8_3(s^-, t^+, u^+). \end{aligned} \tag{31}$$

For $m \geqslant 4$, since any two components of the link have four crossings, there are $4C_2^m = 4m(m-1)/2! = 2m(m-1)$ crossings. Though the calculation is exceedingly tedious, the method is essentially the same; we can still find the one-to-one correspondence between 2^m GHZ states and the 2^m oriented links. Details of this work will be shown in another paper.

Therefore, from the comparison of the correlation tensor of the 2^m GHZ states for an m-node quantum network on the one hand, and the ARM of the 2^m oriented m-component links with $2m(m-1)$ crossings on the other hand, we have found that there is a one-to-one correspondence between the 2^m GHZ states and the 2^m oriented links; thus we give the entangled state a knotted picture, which is very useful in handling and analysing entangled states. We have found that the knotted picture is very useful in discussion of the degree of entanglement, the problem of quantum teleportation and other related phenomema; all these problems we shall systematically discuss in other papers.

References

[1] Einstein A, Podolsky B and Rosen N 1935 *Phys. Rev.* **47** 777
[2] Bennett C H 1993 *Phys. Rev. Lett.* **70** 1895
[3] Ekert A and Josza R 1996 *Rev. Mod. Phys.* **68** 733
[4] DiVincenzo D P 1995 *Science* **270** 255
[5] Huches C H *et al* 1997 *Phys. Rev.* A **56** 1163
[6] Horodecki M and Horodecki P 1999 *Phys. Rev.* A **59** 4206
[7] Cerf N J, Adami C and Gingrich R M 1999 *Phys. Rev.* A **60** 898
[8] Hardy L and Song D D 2000 *Phys. Rev.* A **62** 052315
Fan H 2001 *Phys. Lett.* A **286** 81
[9] Thapliyal A V 1999 *Phys. Rev.* A **59** 3336
[10] Vedral V *et al* 1997 *Phys. Rev. Lett.* **78** 2275
[11] Horodecki P and Lewenstein M 2000 *Phys. Rev. Lett.* **85** 2657
[12] Kauffman L H 1993 *Knots and Physics* 2nd edn (Singapore: World Scientific)
Kleinert H 1990 *Path Integrals in Quantum Mechanics and Polymer Physics* (Singapore: World Scientific)
[13] Yang C N and Ge M L (ed) 1989 *Braid Group, Knot Theory and Statistical Mechanics* (New Jersey: World Scientific)
[14] Qian S W and Gu Z Y 2001 *Commun. Theor. Phys. (Beijing)* **35** 431
[15] Mahler G and Weberrufz V A 1998 *Quantum Networks* 2nd edn (revised and enlarged) (Berlin: Springer)
[16] Bennett C H *et al* 1996 *Phys. Rev.* A **54** 3824